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Note

Lattice path encodings in a combinatorial proof of a differential identity

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Abstract

We specify procedures by which Łukasiewicz paths can encode combinatorial objects, such as involutions, partitions, and permutations. As application, we use these encoding procedures to give a combinatorial proof of the differential operator identity

$$\exp\left(y\left(\frac{\mathrm{d}}{\mathrm{d}x}+f(x)\right)\right)=\exp\left(\int_0^y f(t+x)\mathrm{d}t\right)\exp\left(y\frac{\mathrm{d}}{\mathrm{d}x}\right),$$

due to Stanley. Taylor's theorem is a special case of this differential identity where $f(x)=0$.

Keywords: Lattice paths; Motzkin paths; Łukasiewicz paths; Differential operators

1. Introduction

In this article, the author demonstrates a method of constructing certain well-known combinatorial objects – derangements, partitions, permutations – using weighted one-dimensional lattice paths, in a way that the number of the objects which can be constructed by a path is equal to the weight of that path. Constructions with such properties using Motzkin paths (one-dimensional excursions on non-negative integers where the allowed steps are up by one, down by one, or remain in place) was used by Flajolet [3] to derive continued fraction expansions for series involving factorial numbers, Euler numbers, Eulerian numbers, Stirling numbers of the first kind, and other quantities which could be represented by such combinatorial objects as would lend themselves to his method of construction. In contrast, the author uses Łukasiewicz paths, which generalize Motzkin paths by allowing the up steps to be arbitrarily large. The construction differs from Flajolet's even when restricted to the Motzkin paths, as demonstrated in Section 2.

The upshot of using these generalized lattice paths is that the free Łukasiewicz paths (those that are not restricted to excursions, and indeed have no restrictions on their starting and ending integers) naturally represent monomials of a differential operator of a single variable, given particular conditions on their weights. The idea of representing differential operator monomials as paths is, of course, not new: see, for example, [1,2,4]; the author shows a

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canonical representation of differential operator monomials by Łukasiewicz paths in Section 3. The author's particular construction of partitions and permutations leads to a way of representing certain combinatorial quantities like the Bell numbers and the factorial numbers in terms of differential operators.

Finally, the connection of weighted Łukasiewicz paths, on the one hand, to monomials of differential operators of one variable, and on the other hand, to permutations via construction using the same weights, allows for a combinatorial proof of a differential identity:

$$\exp\left(y\left(\frac{\mathrm{d}}{\mathrm{d}x}+f(x)\right)\right)=\exp\left(\int_0^y f(t+x)\mathrm{d}t\right)\exp\left(y\frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{1}$$

with generic power series $f(x)$, due to Stanley [6] in the context of r-differential posets. Taylor's theorem, $g(x+y)=\exp(y\frac{\mathrm{d}}{\mathrm{d}x})g(x)$, is a special case of this identity, where $f(x)=0$. In Section 4, the author demonstrates the gist of the proof of the generic differential identity (1) via Proposition 5, using the special case where $f(x)=\sum_{j\geq 0}x^j=\frac{1}{1-x}$, since the idea for the general case is similar up to the weights on the steps of the paths.

2. Encoding combinatorial objects by Łukasiewicz paths

We restrict ourselves to paths on non-negative integers known as Łukasiewicz paths: paths which have no restrictions on the up steps $i\to i+j$, $j=0,1,2,\ldots$, but where the down steps must be $i\to i-1$.

Definition 1. A free Łukasiewicz path is a path on the non-negative integers composed of down steps $i\to i-1$, and up steps $i\to i+j$, where j can be any non-negative integer. We assign the weight d_i to each down step $i\to i-1$ and the weight u_i^j to each up step $i\to i+j$. The weight of a free Łukasiewicz path ending at height k is the product of the weights of its steps times x^k.

A Łukasiewicz path is a free Łukasiewicz path which begins and ends at 0, and the weight of a Łukasiewicz path is the product of the weights of its steps.

The next three subsections are examples of procedures for constructing a particular kind of combinatorial object, given a Łukasiewicz path. With respect to a particular construction procedure, we say that a path p *encodes* an object σ if σ can be constructed using p. The weight of the path p must equal to the number of objects p encodes (and, more generally, to the sum of the weights of the objects p encodes).

2.1. Motzkin paths encode involutions

A Motzkin path is a Łukasiewicz path composed only of up steps $i\to i+1$, constant steps $i\to i$, and down steps $i\to i-1$. In this example, we show how a Motzkin path of length n encodes an involution on n elements. The construction presented here differs from the one Flajolet gives in [3]. We consider each step of the path in turn. Suppose we are considering the kth step. If it is the constant step, we create a new singleton (k); if it is the up step, we create a new transposition $(k\ \llcorner\!\lrcorner)$, where we use the character $\llcorner\!\lrcorner$ – a "blank" – to denote a placeholder to be filled later; if it is a down step, we replace any one $\llcorner\!\lrcorner$ in our construction by k.

For the weight of the path to equal the number of the objects it encodes, the weights of the paths would have to be $u_i^0=1$, $u_i^1=1$, and $d_i=i$, since there is only one way to perform the construction for the up step or the constant step, but i ways to perform the construction for the down step $i\to i-1$.

We summarize the construction procedure in a table:

kth step	Construction	Weight factor
$i\to i$	Create a new singleton (k)	$u_i^0=1$
$i\to i+1$	Create a new cycle $(k\ \llcorner\!\lrcorner)$	$u_i^1=1$
$i\to i-1$	Replace any one $\llcorner\!\lrcorner$ with k	$d_i=i$

For example, the Motzkin path $p=0\to 1\to 2\to 1\to 1\to 0$ composed of five steps encodes the transposition (1 5) (2 3) (4) of five elements as follows.

k	kth step	Construction
1	$0 \to 1$	$(1\ \ \llcorner\!\lrcorner)$
2	$1 \to 2$	$(1\ \ \llcorner\!\lrcorner)\ \ (2\ \ \llcorner\!\lrcorner)$
3	$2 \to 1$	$(1\ \ \llcorner\!\lrcorner)\ \ (2\ \ 3)$
4	$1 \to 1$	$(1\ \ \llcorner\!\lrcorner)\ \ (2\ \ 3)\ \ (4)$
5	$1 \to 0$	$(1\ \ 5)\ \ (2\ \ 3)\ \ (4)$

The only other transposition p encodes is $(1\ \ 3)\ \ (2\ \ 5)\ \ (4)$. The number of transpositions encoded by p thus corresponds to the weight of p: $\mathrm{wt}(p) = u_0^1 u_1^1 d_2 u_i^0 d_1 = 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 = 2$.

2.2. *Łukasiewicz paths encode permutations*

The Łukasiewicz paths of length n encode permutations of n objects, using a similar construction procedure. We consider each step of the path in turn, constructing the permutation as we go along. On the kth step, if it is an up step $i \to i + j$, we create a new cycle $(k\ \llcorner\!\lrcorner\ \ldots\ \llcorner\!\lrcorner)$ with j number of blanks; if it is a down step $i \to i - 1$, we replace any one $\llcorner\!\lrcorner$ in our construction by k.

While there is only one way to do the construction for the up step, there are i choices for carrying out the construction of the down step $i \to i - 1$, so the appropriate weights on the steps are $u_i^j = 1$ and $d_i = i$. The construction procedure is summarized as follows.

kth step	Construction	Weight factor
$i \to i + j$	Create a new cycle $(k\ \underbrace{\llcorner\!\lrcorner\ \ldots\ \llcorner\!\lrcorner}_{j})$	$u_i^j = 1$
$i \to i - 1$	Replace any one $\llcorner\!\lrcorner$ with k	$d_i = i$

As an example, the reader is invited to check that the Łukasiewicz path $0 \to 3 \to 2 \to 2 \to 1 \to 2 \to 1 \to 0$ encodes the permutation (1 7 2 4) (3) (5 6).

2.3. *Łukasiewicz paths encode partitions*

To further demonstrate appropriate choice of weights, we show how the Łukasiewicz paths of length n encode partitions of n objects. The construction procedure is exactly the same as for permutations, except for the weight factor u_i^j of the up step $i \to i + j$.

kth step	Construction	Weight factor
$i \to i + j$	Create a new set $\{k\ \underbrace{\llcorner\!\lrcorner\ \ldots\ \llcorner\!\lrcorner}_{j}\}$	$u_i^j = 1/j!$
$i \to i - 1$	Replace any one $\llcorner\!\lrcorner$ with k	$d_i = i$

For example, it is easy to check that the Łukasiewicz path $0 \to 3 \to 4 \to 3 \to 2 \to 2 \to 3 \to 2 \to 1 \to 0$ encodes the partition $\{1\ \ 3\ \ 7\ \ 9\}\ \ \{3\ \ 4\}\ \ \{5\,\}\ \ \{6\ \ 8\}$.

The choice of weights is easily explained through the nature of the construction: if the kth step is the up step $i \to i + j$, there is only one way to perform the assigned construction: create a new set

$$\{k\ \underbrace{\llcorner\!\lrcorner\ \ldots\ \llcorner\!\lrcorner}_{j}\}.$$

However, we take into account that, while the first element in the set is the smallest, there is no guarantee of any sort of order on the other j elements, yet any permutation of them would still describe the same set. So for each set of size $j+1$, there are $j!$ ways in which the same partition can be constructed. With the weights on the steps set as $u_i^j = 1/j!$ and $d_i = i$, the weight of any particular Łukasiewicz path is the number of distinct partitions it encodes.

3. Differential operators as paths

We consider the algebra of differential operators generated by $\frac{\mathrm{d}}{\mathrm{d}x}$ and the operators A_j acting as multiplication by $a_j x^j$ on the ring of polynomials in x over the real numbers, where $j = 0, 1, 2, \ldots$, and a_j are real constants. The actions by these operators can be canonically interpreted in terms of free Łukasiewicz paths, as follows.

3.1. Differential operator monomials as weights of Łukasiewicz paths

The operators $\frac{\mathrm{d}}{\mathrm{d}x}$ and A_j are linear, so they are determined by their actions on the monomials $1, x, x^2, \ldots$, which form a basis of the ring of polynomials in x. Since $\frac{\mathrm{d}}{\mathrm{d}x}(x^i) = ix^{i-1}$, the operator $\frac{\mathrm{d}}{\mathrm{d}x}$ applied to x^i acts as the down step $i \to i-1$ on the set of non-negative integers which correspond to the powers of x, with the weight $d_i = i$. Similarly, $A_j(x^i) = a_j x^{i+j}$, so the operator A_j applied to x^i acts as the up step $i \to i+j$, with the weight $u_i^j = a_j$.

Example 2. If we set $u_i^j = 1, d_i = i$, the weight of the path $2 \to 5 \to 4 \to 3 \to 3 \to 7$ is equal to

$$A_4 A_0 \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x} A_3(x^2) = x^4 \cdot 1 \cdot \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}(x^3 \cdot x^2) = 5 \cdot 4 \cdot x^7.$$

We write the operators corresponding to the steps from right to left, in the order of application; since the path begins at height 2, we apply the composition to x^2.

Example 3. Set $u_i^j = 1, d_i = i$. Then the sum of the weights of all free Łukasiewicz paths with n steps, starting at height k, is

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{1-x}\right)^n (x^k),$$

since each term in $(\frac{\mathrm{d}}{\mathrm{d}x} + 1 + x + x^2 + \cdots)^n$ is a composition of n elements from the set $\{\frac{\mathrm{d}}{\mathrm{d}x}, 1, x, x^2, \ldots\}$.

More generally, we have the following.

Proposition 4. *For a given $f(x) = \sum_{j \geq 0} a_j x^j$, set $u_i^j = a_j, d_i = i$. Then the sum of the weights of all free Łukasiewicz paths with n steps, starting at height k, is*

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + f(x)\right)^n (x^k).$$

In particular, the sum of the weights of all Łukasiewicz paths with n steps is

$$\left.\left(\frac{\mathrm{d}}{\mathrm{d}x} + f(x)\right)^n (1)\right|_{x=0}.$$

3.2. Representing combinatorial numbers in terms of differential operators

By Proposition 4, objects encoded by Łukasiewicz paths can be counted using differential operators, as long as the weight of the down step is $d_i = i$. The weights u_i^j of the up steps determine the function $f(x)$. For example, we construct permutations of n elements from Łukasiewicz paths composed of n steps, with the weights set to $u_i^j = 1, d_i = i$. Therefore, the number of permutations on n elements is

$$n! = \left.\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{1-x}\right)^n (1)\right|_{x=0}.$$

Similarly, we construct partitions of n elements from Łukasiewicz paths composed of n steps, with the weights set to $u_i^j = 1/j!, d_i = i$. The number of partitions of n elements, known as the nth Bell number B_n, is then

$$B_n = \left.\left(\frac{\mathrm{d}}{\mathrm{d}x} + e^x\right)^n (1)\right|_{x=0}.$$

4. Combinatorial proof of the differential identity

As an application of the encoding procedure of permutations by Łukasiewicz paths, we give a combinatorial proof of the differential identity (1) due to Stanley [6], which appears in the context of r-differential posets. The combinatorial proof of the generic differential identity is best illuminated through the special case where $f(x) = \sum_{j\geq 0} x^j = \frac{1}{1-x}$.

Proposition 5.

$$\exp\left(y\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{1-x}\right)\right) = \frac{1-x}{1-x-y} \exp\left(y\frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{2}$$

Proof. We use the encoding procedure of permutations by Łukasiewicz paths to demonstrate the identity (2) of differential operators.

Since differential operators are linear, we consider the action of the left-hand side of Eq. (2) on the monomials. The coefficient of $\frac{y^n}{n!}$ acting on x^k is

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{1-x}\right)^n (x^k).$$

From Example 3, we know that this is the sum of the weights of free Łukasiewicz paths composed of n steps starting at height k, where the weights on the steps are $u_i^j = 1, d_i = i$.

The encoding procedure of permutations by Łukasiewicz paths requires these same weights on the steps:

kth step	Construction	Weight factor
$i \to i+j$	Create a new cycle $(k\ \underbrace{\llcorner\!\lrcorner \ldots \llcorner\!\lrcorner}_{j})$	$u_i^j = 1$
$i \to i-1$	Replace any $\llcorner\!\lrcorner$ with k	$d_i = i$

To use this procedure given a free Łukasiewicz path composed of n steps beginning at height k, since we want the number of $\llcorner\!\lrcorner$ at any step to equal to the current height, we begin with a list of k blanks:

$$[\underbrace{\llcorner\!\lrcorner\ \llcorner\!\lrcorner \ldots \llcorner\!\lrcorner}_{k}].$$

By assigning the weight x to the elements $\llcorner\!\lrcorner$, the weight of a free Łukasiewicz path is then equal to the sum of the weights of the objects it constructs.

The object that is constructed from a free Łukasiewicz path composed of n steps beginning at height k is a pair, consisting of a list of k elements in the set $[n] \cup \{\llcorner\!\lrcorner\}$, and a set of cycles of any size with elements in $[n] \cup \{\llcorner\!\lrcorner\}$. The object has the conditions that the elements in $[n]$ must appear exactly once, and each cycle must contain at least one integer. We will refer to such an object as a *list-permutation pair*.

Let $p_{k,n}$ be the sum of the weights of all list-permutation pairs constructed by free Łukasiewicz paths composed of n steps starting at height k. By construction, $p_{k,n}$ is equal to the sum of the weights of free Łukasiewicz paths composed of n steps starting at height k, so

$$\begin{aligned} F(y) &:= \sum_{n\geq 0} p_{k,n} \frac{y^n}{n!} \\ &= \sum_{n\geq 0} \left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{1-x}\right)^n (x^k) \frac{y^n}{n!} \\ &= \exp\left(y\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{1-x}\right)\right)(x^k). \end{aligned}$$

On the other hand, since the object is a pair composed of a list and a set of cycles, the exponential generating function of the objects is $F(y) = G(y)\exp(H(y))$, where $G(y)$ is the exponential generating function of the list of k

elements, and $H(y)$ is the exponential generating function of the cycle [5]. Each element in the list is either a $\llcorner\!\lrcorner$ or an integer, so $G(y) = (x+y)^k$. Each cycle must have at least one integer, and a cycle can be of any size greater than or equal to 1, so

$$\begin{aligned} H(y) &= \sum_{i\geq 1} \frac{1}{i}((x+y)^i - x^i) \\ &= \sum_{i\geq 1} \frac{(x+y)^i}{i} - \sum_{i\geq 1} \frac{x^i}{i} \\ &= -\ln(1-x-y) + \ln(1-x) \\ &= \ln \frac{1-x}{1-x-y}. \end{aligned}$$

So,

$$\begin{aligned} F(y) &= \exp\left(\ln \frac{1-x}{1-x-y}\right)(x+y)^k \\ &= \frac{1-x}{1-x-y}(x+y)^k, \end{aligned}$$

and therefore

$$\exp\left(y\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{1-x}\right)\right)(x^k) = \frac{1-x}{1-x-y}(x+y)^k.$$

If $g(x)$ is a power series in x, then by linearity

$$\exp\left(y\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{1-x}\right)\right)g(x) = \frac{1-x}{1-x-y}g(x+y).$$

By Taylor's theorem, $g(x+y) = \exp(y\frac{\mathrm{d}}{\mathrm{d}x})g(x)$, so we get the identity of differential operators

$$\exp\left(y\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{1-x}\right)\right) = \frac{1-x}{1-x-y}\exp\left(y\frac{\mathrm{d}}{\mathrm{d}x}\right). \quad \square$$

Note that the proof of Proposition 5 calls on Taylor's theorem. However, Taylor's theorem itself is a special case of the generic identity (1), and the proof of Proposition 5 can be adapted to give a combinatorial proof of Taylor's theorem. By setting the weight of the up steps u_i^j to 0, the exponential generating function $F(y)$ of the Łukasiewicz paths starting at height k becomes

$$F(y) = \exp\left(y\frac{\mathrm{d}}{\mathrm{d}x}\right)(x^k),$$

while by construction, the only objects with non-zero weights that these paths encode are the lists of k elements in the set $[n] \cup \{\llcorner\!\lrcorner\}$ – with the blanks weighted by x and the elements of $[n]$ weighted by y – whose generating function is simply $(x+y)^k$. Thus,

$$\exp\left(y\frac{\mathrm{d}}{\mathrm{d}x}\right)(x^k) = (x+y)^k,$$

and by linearity, we get Taylor's theorem:

$$\exp\left(y\frac{\mathrm{d}}{\mathrm{d}x}\right)g(x) = g(x+y).$$

The combinatorial proof of the generic identity (1) essentially follows the proof of Proposition 5, accounting for the coefficients of $f(x) = \sum_{j\geq 0} f_j x^j$ as weights on the up steps and the corresponding cycles in the list-permutation pairs.

Theorem 6. *Let* $f(x) = \sum_{j \geq 0} f_j x^j$. *Then*

$$\exp\left(y\left(\frac{\mathrm{d}}{\mathrm{d}x} + f(x)\right)\right) = \exp\left(\int_0^y f(t+x)\mathrm{d}t\right)\exp\left(y\frac{\mathrm{d}}{\mathrm{d}x}\right) \tag{3}$$

as differential operators on polynomials in x.

Proof. The left-hand side of Eq. (3) applied to x^k is the exponential generating function of free Łukasiewicz paths starting at height k, where the weights on the steps are $u_i^j = f_j, d_i = i$.

Following the proof of Proposition 5, it is sufficient to re-derive the exponential generating function $F(y)$ of the list-permutation pairs encoded by Łukasiewicz paths starting at height k, where the weight f_{j-1} to each cycle with j elements (since the up step $i \to i + j$ constructs a cycle of size $j + 1$). From Stanley [5], we know that $F(y) = G(y)\exp(H(y))$, where $G(y) = (x + y)^k$ is the exponential generating function of the list of k elements, and $H(y)$ is the exponential generating function of the cycle. Each cycle must have at least one integer, and a cycle of size $i \geq 1$ is weighted by f_{i-1}, so

$$\begin{aligned} H(y) &= \sum_{i \geq 1} \frac{f_{i-1}}{i}((x+y)^i - x^i) \\ &= \sum_{i \geq 1} \frac{f_{i-1}(x+y)^i}{i} - \sum_{i \geq 1} \frac{f_{i-1}x^i}{i} \\ &= \int_0^{x+y} f(t)\mathrm{d}t - \int_0^x f(t)\mathrm{d}t \\ &= \int_0^y f(t+x)\mathrm{d}t. \end{aligned}$$

Thus,

$$F(y) = \exp\left(\int_0^y f(t+x)\mathrm{d}t\right)(x+y)^k,$$

and therefore

$$\exp\left(y\left(\frac{\mathrm{d}}{\mathrm{d}x} + f(x)\right)\right)(x^k) = \exp\left(\int_0^y f(t+x)\mathrm{d}t\right)(x+y)^k.$$

The equality (3) follows by linearity and Taylor's theorem. $\square$

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